Local complementation rule for continuous-variable four-mode unweighted graph states

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The local complementation rule is applied for continuous-variable (CV) graph states in the paper, which is an elementary graph transformation rule and successive application of which generates the orbit of any graph states. The corresponding local Gaussian transformations of local complementation for four-mode unweighted graph states were found, which do not mirror the form of the local Clifford unitary of qubit exactly. This work is an important step to characterize the local Gaussian equivalence classes of CV graph states.

Entanglement lies at the heart of quantum mechanics and plays a crucial role in quantum information processing. Recently, special types of multipartite entangled states, the so-called the graph states [1, 2], have moved into the center of interest. A graph quantum state is described by a mathematical graph, i.e. a set of vertices connected by edges. A vertex represents a physical system, e. g. a qubit (2-dimensional Hilbert space), qudit (d-dimensional Hilbert space), or CV (continuous Hilbert space). An edge between two vertices represents the physical interaction between the corresponding systems. An interesting feature is that many entanglement properties of graph states are closely related to their underlying graphs. They not only provide an efficient model to study multiparticle entanglement [1], but also find applications in quantum error correction [3, 4], multi-party quantum communication [5] and most prominently, serve as the initial resource in one-way quantum computation [6]. Considerable efforts have been stepped toward generating and characterizing cluster state with linear optics experimentally [7, 8, 9, 10]. The principle feasibility of one-way quantum computing model has been experimentally demonstrated through photon cluster state successfully [7, 10].

Most of the concepts of quantum information and computation have been initially developed for discrete quantum variables, in particular two-level or spin-$\frac{1}{2}$ quantum variables (qubits). In parallel, quantum variables with a continuous spectrum, such as the position and momentum of a particle or amplitude and phase quadrature of an electromagnetic field, in informational or computational processes have attracted a lot of interest and appears to yield very promising perspectives concerning both experimental realizations and general theoretical insights [11, 12], due to relative simplicity and high efficiency in the generation, manipulation, and detection of CV state. Although up to six-qubit single-photon cluster states have been created via postselection using nonlinear and linear optics, the deterministic, unconditional realization of optical cluster states would be based on continuous variables. CV cluster and graph states have been proposed [13], which can be generated by squeezed state and linear optics [14, 15], and demonstrated experimentally for four-mode cluster state [16]. The one-way CV quantum computation was also proposed with CV cluster state [17]. Moreover, the protocol of CV anyonic statistics implemented with CV graph states is proposed [18].

One of the interesting issues on entanglement is how to define the equivalence of two entangled states. The transformations of qubit graph states under local Clifford operations were studied by Hein [1] and Van den Nest [19]. They translate the action of local Clifford operations on qubit graph states into transformations on their associated graphs, that is, to derive transformations rules called the local complement rule, stated in purely graph theoretical terms, which completely characterize the evolution of graph states under local Clifford operations. The corresponding local Clifford unitary is a single and simple form. The successive application of this rule suffices to generate the complete orbit of any qubit graph state under local Clifford operations. In this paper, the local complement rule for CV four-mode unweighted graph state is applied and the corresponding local Clifford transformations (also called local Gaussian transformation for CV) for four-mode graph state were found. The local Gaussian equivalence classes of CV four-mode unweighted graph states can be obtained by this way. It was shown that the corresponding local Gaussian unitary can not exactly mirror that for qubit, which is not a single form compared with qubit. This result shows the complexity of CV quantum systems and stimulate the research on the local Gaussian equivalence of CV graph states. Although only focusing on the CV four-mode unweighted graph states, this work makes an important step in the direction of addressing the general question "What are the graph transformation rules that describe local unitary equivalence of any CV graph states?".

The CV operations are reviewed firstly that follow the standard prescription given in Ref.[20]. The Pauli X and Z operators of qubit are generalized to the Weyl-Heisenberg group, which is the group of phase-space displacements. For CVs, this is a Lie group with generators $\hat{x} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$ (quadrature-amplitude or position) and $\hat{p} = -i(\hat{a} - \hat{a}^\dagger)/\sqrt{2}$ (quadrature-phase or momentum) of the electromagnetic field as the CV system. These operators satisfy the canonical commutation relation $[\hat{x}, \hat{p}] = i$ (with $\hbar = 1$). In analogy to the qubit Pauli operators, the single mode Pauli operators are defined as $X(s) = exp[-is\hat{p}]$ and $Z(t) = exp[it\hat{x}]$ with $s, t \in \mathbb{R}$. The Pauli operator $X(s)$ is a position-translation operator, which acts on the computational basis of position eigenstates $\{|q\rangle; q \in \mathbb{R}\}$ as $X(s)|q\rangle = |q+s\rangle$, whereas Z is a momentum-translation operator, which acts on the momentum eigenstates as $Z(t)|p\rangle = |p + t\rangle$. These operators are non-commutative

and obey the identity $X(s)Z(t) = e^{-ist}Z(t)X(s)$. The Pauli operators for one mode can be used to construct a set of Pauli operators $\{X_i(s_i), Z_i(t_i); i = 1, ..., n\}$ for n-mode systems. This set generates the Pauli group $\mathcal{C}_1$. The clifford group $\mathcal{C}_2$ is the normalizer of the Pauli group, whose transformations acting by conjugating, preserve the Pauli group $\mathcal{C}_1$; i.e., a gate U is in the Clifford group if $URU^{-1} \in \mathcal{C}_1$ for every $R \in \mathcal{C}_1$. The clifford group $\mathcal{C}_2$ for CV is shown [20] to be the (semidirect) product of the Pauli group and linear symplectic group of all one-mode and two-mode squeezing transformations. Transformation between the position and momentum basis is given by the Fourier transform operator $F = exp[i(\pi/4)(\hat{x}^2 + \hat{p}^2)]$, with $F|q\rangle_x = |q\rangle_p$. The action FRF^{-1} of the Fourier transform on the Pauli operators is

$$\begin{aligned} F : X(s) &\rightarrow Z(s), \\ Z(t) &\rightarrow X(-t). \end{aligned} \tag{1}$$

This is the generalization of the Hadamard gate for qubits. The phase gate $P(\eta) = exp[i(\eta/2)\hat{x}^2]$ with $\eta \in \mathbb{R}$ is a squeezing operation for CV and the action on the Pauli operators is

$$\begin{aligned} P(\eta) : X(s) &\rightarrow e^{-is^2\eta/2}Z(s\eta)X(s), \\ Z(t) &\rightarrow Z(t), \end{aligned} \tag{2}$$

in analogy to the phase gate of qubit [21]. The controlled operation C-Z is generalized to controlled-$Z(C_Z)$. This gate $C_Z = exp[i\hat{x}_1 \bigotimes \hat{x}_2]$ provides the basic interaction for two mode 1 and 2, and describes the quantum nondemolition (QND) interaction. This set $\{X(s), F, P(\eta), C-Z; s, \eta \in \mathbb{R}\}$ generates the Clifford group. Transformations in the Clifford group do not form a universal set of gates for CV quantum computation. However, Clifford group transformation (Gaussian transformations) together with any higher-order nonlinear transformation (non-Gaussian transformation) acting on a single-mode form a universal set of gates [20]. The local Gaussian group only was concerned here, which can be obtained by repeated application of Fourier and phase gates. In the following, another type of the phase gate will be used $P_X(\eta) = FP(\eta)F^{-1} = exp[i(\eta/2)\hat{p}^2]$ and the action on the Pauli operators is

$$\begin{aligned} P_X(\eta) : X(s) &\rightarrow X(s), \\ Z(t) &\rightarrow e^{-it^2\eta/2}X(-t\eta)Z(t), \end{aligned} \tag{3}$$

where $P_X(\eta)^\dagger = P_X(\eta)^{-1} = P_X(-\eta)$.

A graph quantum state is described by a mathematical graph $G = (V, E)$, i.e. a finite set of n vertices V connected by a set of edges E [22]. An $\{a, c\}$-path is a order list of vertices $a = a_1, a_2, \ldots, a_{n-1}, a_n = c$, such that for all i, a_i and a_{i+1} are adjacent. A connected graph is a graph that has an $\{a, c\}$-path for any two $a, c \in V$. Otherwise it is referred to as disconnected. The neighborhood $N_a \subset V$ is defined as the set of vertices b for which $\{a, b\} \in E$. When a vertex a is deleted in a graph G, together with all edges incident with a, one obtains a new graph, denoted by $G - a$. For a subset of vertices

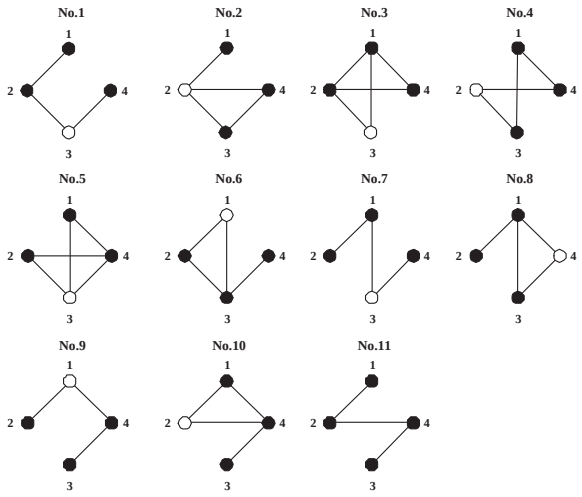

FIG. 1: The connected four-vertex graphs for an successive application of the local complementation. The rule is successively applied to the vertex, which is circle in the figure.

$U \subset V$ of a graph $G = (V, E)$ let us denote with $G - U$ the graph that is obtained from G by deleting the set U of vertices and all edges which are incident with an element of U. Similarly, an subgraph $G[C]$ of a graph $G = (V, E)$, where $C \subset V$, is obtained by deleting all vertices and the incident edges that are not contained in C. The preparation procedure of CV graph states [13] can exactly mirror that for qubit graph states only using the Clifford operations: first, prepare each mode (or graph vertex) in a phase-squeezed state, approximating a zero-phase eigenstate (analog of Pauli-X eigenstates), then, apply a QND interaction (C-Z gate) to each pair of modes (j, k) linked by an edge in the graph. All C-Z gates commute. Thus, the resulting CV graph state becomes, in the limit of infinite squeezing, $g_a = (\hat{p}_a - \sum_{b \in N_a} \hat{x}_b) \rightarrow 0$, where the modes $a \in V$ correspond to the vertices of the graph of n modes, while the modes $b \in N_a$ are the nearest neighbors of mode a. This relation is as a simultaneous zero-eigenstate of the position-momentum linear combination operators. The stabilizers $G_a(\xi) = exp[-i\xi g_a] = X_a(\xi)\prod_{b \in N_a} Z_b(\xi)$ with $\xi \in \mathbb{R}$ for CV graph states are analogous to n independent stabilizers $G_a = X_a \prod_{b \in N_a} Z_b$ for qubit graph states. Note that the CV graph states that is discussed here are unweighted since the QND interactions all have the same strength. For the CV weighted graph states generated by the different QND interaction strength, the stabilizers become $G_a(\xi) = X_a(\xi)\prod_{b \in N_a} Z_b(\Omega_{ab}\xi)$, where Ω_{ab} is the interaction strength between mode a and b. The CV weighted graph states are more complex, which is not considered in this paper.

The action of the local complement rule, can be described as: letting $G = (V, E)$ be a graph and $a \in V$ be a vertex, the local complement of G for a, denoted by $\lambda_a(G)$, is obtained by complementing the subgraph of G generated by the neighborhood N_a of a and leaving the rest of the graph unchanged.

The successive application of this rule suffices to generate the complete orbit of any graph. Here, the corresponding local Gaussian unitary for CV four-mode graph state were examined. The corresponding four-mode graph state $|\lambda_a(G)\rangle$ by local complement of a graph G at some vertex $a \in V$, is given by a local Gaussian unitary operation

$$|\lambda_a(G)\rangle = U_{\lambda_a}|G\rangle, \tag{4}$$

where U_{λ_a} is local Gaussian operation. A form of the local Gaussian unitary comprising two types of phase gate is defined

$$U_{LG_a} = P_{Xa}(1) \prod_{b \in N_a} P_b(-1), \tag{5}$$

which mirrors the form of qubit local Clifford operation for local complementation. Fig.1 depicts connected four-mode graphs by such a successive application of the local complement rule. The four independent stabilizers of the first graph state $|G^{(1)}\rangle$ are given by

$$\begin{aligned} G_1^{(1)}(\xi) &= X_1(\xi)Z_2(\xi), \\ G_2^{(1)}(\xi) &= X_2(\xi)Z_1(\xi)Z_3(\xi), \\ G_3^{(1)}(\xi) &= X_3(\xi)Z_2(\xi)Z_4(\xi), \\ G_4^{(1)}(\xi) &= X_4(\xi)Z_3(\xi). \end{aligned} \tag{6}$$

with $G_i^{(1)}(\xi)|G^{(1)}\rangle = |G^{(1)}\rangle$ in the limit of infinite squeezing, where $i = 1, ..., 4$. Applying the local Gaussian unitary U_{LG_3} to the vertex 3, I can compute the four independent stabilizers of the resulting graph state $|G^{(2)}\rangle$ by Eqs. (2,3,4,5), for example calculating $G_2^{(2)}(\xi)$,

$$\begin{aligned} |G^{(2)}\rangle &= |\lambda_3(G^{(1)})\rangle \\ &= U_{LG_3}G_2^{(1)}(\xi)|G^{(1)}\rangle \\ &= U_{LG_3}G_2^{(1)}(\xi)U_{LG_3}^{-1}U_{LG_3}|G^{(1)}\rangle \\ &= [e^{i\xi^2/2}Z_2(-\xi)X_2(\xi)]Z_1(\xi) \\ &\quad [e^{-i\xi^2/2}X_3(-\xi)Z_3(\xi)]U_{LG_3}|G^{(1)}\rangle \\ &= X_2(\xi)Z_1(\xi)Z_3(\xi)U_{LG_3} \\ &\quad [Z_2(-\xi)X_3(-\xi)]|G^{(1)}\rangle \\ &= X_2(\xi)Z_1(\xi)Z_3(\xi)U_{LG_3} \\ &\quad [Z_2(-\xi)X_3(-\xi)]G_3^{(1)}(\xi)|G^{(1)}\rangle \\ &= X_2(\xi)Z_1(\xi)Z_3(\xi)Z_4(\xi)|\lambda_3(G^{(1)})\rangle \\ &= G_2^{(2)}(\xi)|G^{(2)}\rangle \end{aligned} \tag{7}$$

to obtain

$$\begin{aligned} G_1^{(2)}(\xi) &= X_1(\xi)Z_2(\xi), \\ G_2^{(2)}(\xi) &= X_2(\xi)Z_1(\xi)Z_3(\xi)Z_4(\xi), \\ G_3^{(2)}(\xi) &= X_3(\xi)Z_2(\xi)Z_4(\xi), \\ G_4^{(2)}(\xi) &= X_4(\xi)Z_2(\xi)Z_3(\xi), \end{aligned} \tag{8}$$

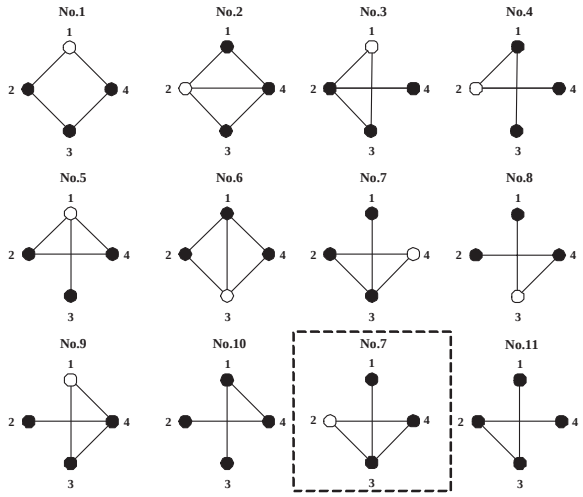

FIG. 2: The set of four-vertex graphs is equivalent to Fig.1 under local Gaussian transformation and graph isomorphisms. The graph No.7, which is repeated and placed in the dash-line box behind the No.10, is used for generating the graph No.11 directly.

which exactly correspond to the stabilizers of No.2 graph state in Fig.1. The complete orbit of the first graph can be obtained by applying the local complement rule repeatedly to the vertices and the corresponding local Gaussian unitary is shown in the following forms: $No.1 \overset{U_{LG_3}}{\longrightarrow} No.2 \overset{U^2_{LG_3}F_1^2U^\dagger_{LG_2}}{\longrightarrow} No.3 \overset{U^\dagger_{LG_3}}{\longrightarrow} No.4 \overset{U_{LG_1}}{\longrightarrow} No.5 \overset{U^2_{LG_2}F_1^2U^\dagger_{LG_3}}{\longrightarrow} No.6 \overset{U^\dagger_{LG_1}}{\longrightarrow} No.7 \overset{U_{LG_3}}{\longrightarrow} No.8 \overset{U^\dagger_{LG_4}}{\longrightarrow} No.9 \overset{U_{LG_1}}{\longrightarrow} No.10 \overset{U^\dagger_{LG_2}}{\longrightarrow} No.11$. Here the complete orbit means the local complement rule is applied on the graph until exhaust all possibilities. Notice the difference in the Gaussian operations of $2 \rightarrow 3$, and $5 \rightarrow 6$. In the qubit case, these would have been of identical form. This shows the added richness of CV graph states over their qubit counterparts. Note that Hein et al. [1] classify the equivalence of the graph states by considering the local complementation and additional graph isomorphisms, which corresponds to the permutations of the vertices. Fig.2 shows another set of graphs, which are not equivalent to any graph in the equivalence class represented in Fig.1 only considering the local complementation. However, they belong to the same equivalence class when considering both, local Gaussian unitary and graph isomorphisms. The corresponding local Gaussian unitary in Fig.2 is shown in the following forms: $No.1 \overset{U_{LG_1}}{\longrightarrow} No.2 \overset{U^2_{LG_3}F_1^2U^\dagger_{LG_2}}{\longrightarrow} No.3 \overset{U^\dagger_{LG_1}}{\longrightarrow} No.4 \overset{U_{LG_2}}{\longrightarrow} No.5 \overset{U^2_{LG_2}F_3^2U^\dagger_{LG_1}}{\longrightarrow} No.6 \overset{U^2_{LG_4}F_2^2U^\dagger_{LG_3}}{\longrightarrow} No.7 \overset{U^\dagger_{LG_4}}{\longrightarrow} No.8 \overset{U_{LG_3}}{\longrightarrow} No.9 \overset{U^\dagger_{LG_1}}{\longrightarrow} No.10; No.7 \overset{U^\dagger_{LG_2}}{\longrightarrow} No.11$.

The set of graphs in Fig.3, usually called GHZ (Greenberger-Horne-Zeilinger) entangled states, is not equivalent with Fig.1 and 2 under local Gaussian transformation and graph isomorphisms. The local Gaussian unitary is applied to four-mode graph states in Fig.3, which is written

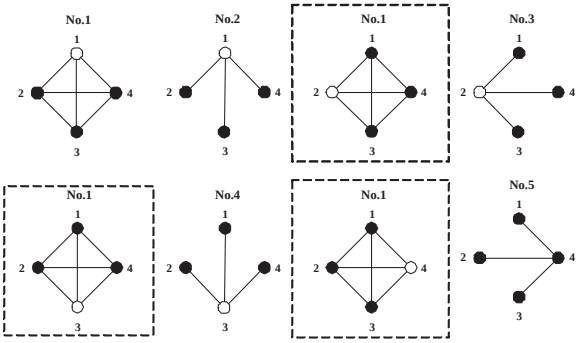

FIG. 3: The set of four-vertex graphs is not equivalent to Fig.1 and 2 under local Gaussian transformation and graph isomorphisms. The graph No.1, which is placed in the dash-line box, is used repeatedly by the local complementation.

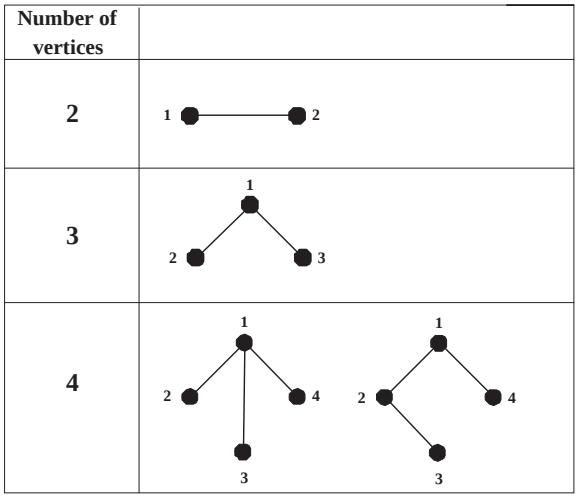

FIG. 4: The connected graphs with up to four vertices are not equivalent under local Gaussian transformation and graph isomorphisms.

above the arrows of the following diagram: $No.1 \overset{U^{\dagger}_{LG_1}}{\longrightarrow} No.2 \overset{U_{LG_1}}{\longrightarrow} No.1 \overset{U^{\dagger}_{LG_2}}{\longrightarrow} No.3 \overset{U_{LG_2}}{\longrightarrow} No.1 \overset{U^{\dagger}_{LG_3}}{\longrightarrow} No.4 \overset{U_{LG_3}}{\longrightarrow} No.1 \overset{U^{\dagger}_{LG_4}}{\longrightarrow} No.5$. Fig. 4 lists the graphs with up to four vertices that are not equivalent under local Gaussian transformation and graph isomorphisms.

In summary, the local complement rule was extended for CV graph states and the corresponding local Gaussian transformations of four-mode unweighted graph states were given. Thus the local Gaussian equivalence classes of CV four-mode unweighted graph states can be obtained. It was shown that the corresponding local Clifford unitary can not exactly mirror that for qubit and demonstrate the complexity of CV quantum systems. It is worth remarking that, whether the local complementation for any CV graph states can be implemented completely by the local Gaussian transformations and the general form of the corresponding local Gaussian unitary can be found, still need be further investigated. This work not only contribute to a deeper and more complete understanding of CV multipartite entanglement, but also stimulate the research on CV graph states theoretically and experimentally.

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